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DISCUSSION OF PROCEEDINGS - SEPARATES

157, 202, 223,

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DISCUSSION OF
RADIAL IMPACK ON AN ELASTICALLY SUPPORTED RING
PROCEEDINGS—SEPARATE NO. 157

JOHN C. BURGESS,[24] J.M. ASCE.—The theoretical development presented by the author essentially follows that given by S. Timoshenko,[25] in that it uses both the elemental and energy approaches. Since the paper is not primarily an exposition of method, and the problem could have been completely solved by either of these approaches, the development seems unduly encumbered. Rather than using the suggestions found in the literature[26] for his normal functions and restricting his development to the energy approach, Mr. Wenk integrated the differential equation of motion. Since the boundary conditions, and not the load, determine the normal functions, Eq. 14 (for the homogeneous case) gives the correct normal functions. To avoid a change in method at this point, Eq. 15*c* can be substituted into Eq. 6*b*, and the load expanded in terms of the same normal function (cos $i\,\theta$). The result is a combination of Eqs. 18*b* and 21 directly, rather than a use of the Lagrange equations of motion.

With regard to notation, confusion is caused by the choice of index in Eqs. 12 and 14. These equations might be more clear if written as follows:

$$u_\theta = \cdots = u_{\theta+2k\pi},\ k = 0,\ 1,\ 2,\ \cdots \qquad (33)$$

and

$$u = \sum_{i=1}^{\infty} (C_i \cos i\,\theta + D_i \sin i\,\theta) \cos (p_i\,t + \delta) \qquad (34)$$

Eqs. 22 and 23 represent "steady-state" solutions only for specific ratios of the forcing and natural frequencies. The solution that usually represents the steady state is one having no terms corresponding to free vibrations. Solutions for vibration problems in which the natural damping is small may be obtained by omitting explicit recognition of damping in the differential equation, and taking account of it implicitly by adjusting the complete solution for the undamped case. The reasoning is that, because of the small natural damping always present in a physical system, the free-vibration terms (or transients) will decrease with time, leaving only the forcing terms.[27] Eq. 22 might therefore be rewritten as

$$a_i = -\frac{P_0\,\alpha\,i_a}{\omega^2 - p^2_i} \sin \omega\,t_1 \qquad (35)$$

with Eqs. 23 changed accordingly.

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[24] Research Asst., Stanford Univ., Stanford, Calif.

[25] "Vibration Problems in Engineering," by S. Timoshenko, D. Van Nostrand, Inc., New York, N. Y., 2d Ed., 1937, Chapter VI.

[26] *Ibid.*, pp. 408-409.

[27] *Ibid.*, p. 350.

EDWARD WENK, JR.,[28] J. M. ASCE.—The careful study undertaken and the suggestions offered by Mr. Burgess are most welcome. Either the elemental or the energy approaches to the problem under consideration would, of course, have been valid. In this type of problem, the choice is usually a matter of personal preference. Any detailed discussion of the merits of various approaches is not within the scope of the paper.

[28] Head, Structures Div., Structural Mech. Lab., David Taylor Model Basin, Washington, D. C.

The primary considerations in using both methods were (1) to arrive at the frequency equation (Eq. 13) with as few arbitrary assumptions as possible, and (2) to develop solutions for response to impact in terms of the normal modes of vibration so as to facilitate the interpretation of any numerical results. The avoidance of the change in method proposed by Mr. Burgess could have been accomplished only by expansion of the load in terms of normal functions. For the concentrated load selected for analysis, this procedure introduces a form of the solution that is too cumbersome for numerical computation.

In suggesting that the free vibration terms of Eqs. 22 and 23 be omitted, Mr. Burgess follows the assumption, made by many theorists, that these vibrations are so rapidly "damped out" as to be unimportant. However, the effects of damping, at least with respect to the lower modes, are not likely to be this severe for the interval during which the impact load is acting on the structure. The importance of these free vibration terms is demonstrated in Figs. 11 and 12 as deviations from the simple sine form, for $t < \tau$. Other more detailed illustrations of this effect are given elsewhere.[29] By neglecting these terms, as proposed by Mr. Burgess, agreement between the theoretical analysis and the experimental results would not have been as good. The loss in accuracy thus introduced could be avoided by using the complete form of the solution, as given in the paper, which in itself is not difficult to solve. Perhaps the importance of retaining these terms in impact problems, contrary to the usual practice,[27] should be emphasized.

[29] "A Theoretical and Experimental Investigation of a Dynamically Loaded Ring with Radial Elastic Support," by E. Wenk, Jr., *TMB Report No. 704*, David Taylor Model Basin, Washington, D. C., July, 1950, pp. 18–21, Figs. 5 and 6.

DISCUSSION OF
THE APPLICATION OF HEAVISIDE'S STEP-FUNCTION
TO BEAM PROBLEMS
PROCEEDINGS—SEPARATE NO. 202

Bruno Poggi.[1]—The application of Heaviside's step-function to the study of deflected beams makes it possible to obtain an elastic curve formula which is valid for the whole beam and for every load condition. This application also has the advantage of restricting the number of the arbitrary constants of integration. This has been shown in Professor Goldberg's paper[2]. It may be of interest to call attention to the close connection between the unit-step function and the unit-impulse function, whose application in the same way is of advantage in beam deflection problems involving discontinuities in load function.

We assume as known that a unit-impulse function, U_a, means a function $F(x)$ which is zero everywhere except in a very small interval δ around a point $(x = a)$ where it takes values whose integral $\int_\delta F(x)\,dx$ is always equal to one. U_a is bound to the step-function by the relationship

$$\int_0^x U_a\,dx = H_a \tag{1}$$

Eq. (1) is obtained by making use of a property of U_a resulting from the definition itself:

$$\int_\alpha^\beta F(x)\,U_a\,dx = \begin{cases} 0 & a < \alpha \quad \text{or} \quad a > \beta \\ F(a) & \alpha \leqslant a \leqslant \beta \end{cases} \tag{2}$$

Substantially the method for the application of U_a to beam problems does not differ from the method used by Professor Goldberg for function H_a; we will herein perform the deduction of a formula for the elastic curve of prismatic beams, generally valid for every loading distribution. The differential equation for the elastic curve which concerns homogeneous prismatic beams is:

$$EI\,\frac{d^4y}{dx^4} = w(x)\,. \tag{3}$$

After integrating Eq. (3) with the Laplace transformation, we obtain:

$$EI\left\{p^4\mathcal{L}[y] - p^3y(0) - p^2y'(0) - p\,y''(0) - y'''(0)\right\} = \mathcal{L}[w(x)]$$

or

$$\mathcal{L}[y] = \frac{1}{EI}\cdot\frac{\mathcal{L}[w(x)]}{p^4} + \frac{y(0)}{p} + \frac{y'(0)}{p^2} + \frac{y''(0)}{p^3} + \frac{y'(0)}{p^4}\,. \tag{4}$$

1. Assistant of Hydraulic Constructions. Associate Professor of Complements of Mathematics. Engineering Faculty—University of Bologna.
2. On the use of step-function studying deflected beams, through the Laplace transformation, see also: R. Tavani: "Sull'impiego della trasformazione di Laplace nello studio delle travi inflesse"—Atti dell'Instituto di Scienza delle Costruzioni dell'Università di Pisa, 1949.

Upon performing inverse transformation of the two sides of Ed. (4), bearing in mind that $\mathcal{L}^{-1}\left[\frac{1}{p^n}\right] = \frac{x^{n-1}}{(n-1)!}$ and specifying initial values with relationships:

$y'(0) = \phi(0)$ rotation at the 0

$EI\, y''(0) = -M(0)$ bending moment at the 0

$EI\, y'''(0) = -V(0)$ shearing force at the 0

we obtain:

$$y(x) = \frac{w(x)}{EI} * \frac{x^3}{6} + y(0) + \phi(0)\, x - \frac{M(0)}{EI} \cdot \frac{x^2}{2} - \frac{V(0)}{EI} \cdot \frac{x^3}{6} \cdot \qquad (5)$$

The first term of the second side of Eq. (5) is the "faltung" of the two functions $\frac{w(x)}{EI}$ and $\frac{x^3}{6}$ and is defined by

$$\frac{w(x)}{EI} * \frac{x^3}{6} = \frac{1}{6EI} \int_0^x (x-\xi)^3\, w(\xi)\, d\xi. \qquad (6)$$

Ed. (6), on the ground of Eq. (2), is applicable even if the function $w(x)$ also includes n impulse functions (concentrated loads) $P_1 U_{a_1}$, $P_2 U_{a_2}, \ldots P_n U_{a_n}$. Therefore

$$w(x) = \overline{w}(x) + \sum_1^n{}_i\ P_i\ U_{a_i}$$

with $\overline{w}(x)$ a continuous function which can be expandet in a power series. Consequently Eq. (5) can be written:

$$y(x) = \frac{1}{6EI} \int_o^x (x-\xi)^3\, \overline{w}(\xi)\, d\xi + \frac{1}{6EI} \int_o^x (x-\xi)^3 \sum_1^n{}_i\ P_i U_{a_i}\ d\xi + \\ + y(0) + \phi(0)\, x - \frac{M(0)}{EI} \cdot \frac{x^2}{2} - \frac{V(0)}{EI} \cdot \frac{x^3}{6} \quad . \qquad (7)$$

Then Eq. (7) can be considered to be the general formula of the elastic curve for a prismatic beam. It should be noted that from Eq. (7), as a singular case, the Maclaurin expansion is obtained, as shown by N. K. Snitco and later by M. Hetényi[3]. In effect supposing

$$w(\xi) = \overline{w}(\xi) = w(0) + w'(0)\, \xi + w''(0)\, \frac{\xi^2}{2} + w'''(0)\, \frac{\xi^3}{6} + \ldots\ldots$$

substituting in Eq. (7), and integrating, one obtains

$$y(x) = y(0) + \phi(0)x - \frac{M(0)}{EI} \cdot \frac{x^2}{2} - \frac{V(0)}{EI} \cdot \frac{x^3}{3!} + \frac{w(0)}{EI} \cdot \frac{x^4}{4!} + \frac{w'(0)}{EI} \cdot \frac{x^5}{5!} + \ldots$$

which is the expansion derived by Snitco-Hetényi.

3. See for further references: O. Belluzzi - Scienza delle Costruzioni - Vol. I, Cap. I, Page 308. N. Zanichelli publisher. Bologna, 1948.

Herbert Saunders.—The author has done an excellent job in showing the application of the Heaviside Step-Function to Beam Problems. The use of the Laplace Transform would be much more suited to the beam problems as there are a number of tables available for use. The advantages of the Laplace Transform are its ease of manipulation and reduction of the tedious effort in arriving at a solution of the problem. Professor Thomsen[1] has gone into great detail in the solution of many beam problems including statically determinate and indeterminate, stepped beams and buckling effects upon columns. Dr. Pipes[2] has applied the Laplace Transform to problems of beams on an elastic foundation with a moderate amount of success. As in communication engineering, the Heaviside method has been replaced by the more adaptable Laplace Transform and would definitely accomplish this fact here with no great difficulty.

1. Wm. Thomsen—Laplace Transform—Prentice Hall Publications (1950).
2. L. A. Pipes—Applied Mathematics for Engineers—McGraw-Hill (1946).

DISCUSSION OF
APPLICATIONS OF THE RELAXATION TECHNIQUE IN FLUID MECHANICS
PROCEEDINGS—SEPARATE NO. 223

Fred W. Blaisdell,[1] M. ASCE—In describing the relaxation technique together with its advantages and its limitations, the authors have provided a distinct service to those who might consider the use of this tool. The examples of the application of the relaxation technique are most interesting and the comparisons with experimental data, where presented, show very good agreement. The writer would like to comment on one of the examples.

The writer recently conducted a study of experimental data obtained by others on the form of the nappe emitting from a vertical, sharp-crested, suppressed weir. A general equation was obtained for that portion of the nappe which was free falling. This conforms to the region defined by the authors where, "Far to the right the streamlines approach the trajectory of free fall, and the internal pressure approaches zero."

The data available to the writer indicates the free fall region to include values of x/H in excess of about 0.5. Therefore, the writer's results and the authors' table should yield comparable results when $x/H = 0.5$. The comparison is made in the accompanying table. For the lower nappe there seems to be some trend to the differences between the writer's and the authors' results. For the upper nappe, the differences are erratic and no trend can be detected.

A second comparison and test can also be applied. The general equation developed by the writer has the form

$$\frac{y}{H} = C + \tan\theta \frac{x}{H} + \frac{S}{2}\left(\frac{x}{H}\right)^2 + \frac{T}{H}$$

The second differential of this equation yields

$$\frac{d^2(y/H)}{d(x/H)^2} = S$$

This indicates that the second differences should be constant. The second differences are given in the accompanying table. The lack of absolute constancy in the writer's second differences is due to the recording of the nappe coordinates to only three significant figures. The writer's value for S is 0.850. There seems to be considerable variation in the second differences derived from the authors' data.

Still a third comparison and test can be applied to the data. The term T in the equation for the nappe form represents the vertical thickness of the nappe. This dimension is a constant because the horizontal velocity is constant where the nappe is free falling. (For the lower nappe the term T is zero.) The nappe thickness in terms of H computed from the authors' data is given in the table. It may be compared with 0.559, the nappe thickness as derived by the writer for the authors' boundary conditions.

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1. Project Supervisor, U. S. Department of Agriculture, SCS, St. Anthony Falls Hydraulic Laboratory, Minneapolis, Minnesota.

The validity of comparing y/H values perhaps is open to some question, since either method of analysis could yield valid results from the basic data and assumptions, and one could perhaps not choose as to which gave the most accurate values. However, the comparison within each group of data of nappe thickness and, more particularly, of the second differences, should yield some information on the precision of the results. Based on the authors' paper, it would seem that the relaxation technique should give identical nappe thicknesses for each value of x/H and identical second differences if the network were sufficiently subdivided. This does not seem to be completely substantiated by the writer's tests of the data.

The authors' comments on these points would be much appreciated.

COMPARISON OF NAPPE SURFACE COORDINATES

$\frac{x}{H}$	Lower Nappe				Upper Nappe				$\frac{T}{H}$
	y/H		$\Delta^2(y/H)/\Delta(x/H)^2$		y/H		$\Delta^2(y/H)/\Delta(x/H)^2$		
	Writer:	Author	Writer:	Author	Writer:	Author	Writer:	Author	Author
0.500	+0.071	+0.065			+0.630	+0.678			
0.625	+0.018	+0.010	-0.832	-0.832	+0.577	+0.609	-0.832	-0.576	
0.750	-0.048	-0.058	-0.832	-0.576	+0.511	+0.531	-0.832	-1.984	
0.875	-0.127	-0.135	-0.896	-0.640	+0.432	+0.422	-0.896	+1.728	0.577
1.000	-0.220	-0.222	-0.832	-0.896	+0.339	+0.340	-0.832	-1.792	0.562
1.125	-0.326	-0.323	-0.832	-0.512	+0.233	+0.230	-0.832	-0.384	0.553
1.250	-0.445	-0.432	-0.896	-1.280	+0.114	+0.114	-0.896	-0.576	0.546
1.375	-0.578	-0.561	-0.832	-0.960	-0.019	-0.011	-0.832	-0.832	0.550
1.500	-0.724	-0.705	-0.832	-1.088	-0.165	-0.149	-0.832	-1.152	0.556
1.625	-0.883	-0.866	-0.832	-1.024	-0.324	-0.305	-0.832	-1.920	0.561
1.750	-1.055	-1.043	-0.896		-0.496	-0.491	-0.896	-1.088	0.552
1.875	-1.241		-0.832		-0.682	-0.694	-0.832	-0.832	
2.000	-1.440				-0.881	-0.910			
Average			-0.849	-0.868			-0.849	-0.855	0.555
								Writer	0.559

H. M. Paynter, J.M. ASCE, and R. F. Scott.--The writers would differ with the definition of relaxation methods given in the paper. Rather than merely being techniques of numerical integration, they express an entire mode of attack on any problem or set of relations, be they graphic, algebraic, or numerical. They constitute much the same approaches which have come to be called "Hardy Cross" methods by American engineers. Essentially they involve the prior determination of the mutual influence patterns among the variables, followed by a systematic successive application of this pattern (stylus or matrix) to each and every equation of the system until the unbalances (residuals) are reduced within the required tolerances.

The general philosophy behind all such methods is discussed fully in a compilation edited by Grinter[3] as well as in an exemplary paper by Emmons[1]. Besides the Southwell treatise mentioned by the authors, further details and special manipulations are treated in the books of Salvadori and Baron[8], Scarborough[9], Milne[6], and Shaw[11].

The relaxation principle differs from simple iteration in that the personal judgment and insight of the computer can enter at each step of the process. Relaxation methods reduce to purely iterative procedures if this judgment factor is removed. This point is clearly brought out in the paper of Emmons, previously cited.

Another most important distinction to make is that between steady state and transient solutions as they affect the applicability of different numerical approaches.

The diffusion equation:

$$\frac{\partial \phi}{\partial t} = a \nabla^2 \phi$$

reduces to the Laplace equation

$$\nabla^2 \phi = 0$$

under steady conditions ($\frac{\partial \phi}{\partial t} = 0$). If, for example, it was desired to find the values of seepage flow through a dam during a drawdown of the reservoir, or the hydrostatic excess pore water pressure values during a soil consolidation process, relaxation methods would not be appropriate. In these cases, iterative techniques, such as those used for thermal diffusion or soil consolidation problems[6], [1], [2], [10] become useful.

In similar manner, when the Poisson equation:

$$\nabla^2 \phi = f(x, y, z)$$

is to be solved, there are certain cases where with the proper choice of grid and symmetry conditions direct numerical solutions are often possible. One of the writers has studied such a case involving the cooling of a chemical reactor in which heat was generated uniformly and removed by a grillage of cooling tubes. Here a hexagonal grid was used, making maximum use of symmetry, and the heat flow could be summed directly by calculating outward from the tubes.

The writers regret that once again in this paper not all of the interesting features and consequences of the potential flow solutions are "tied down", so to speak, similar to the manner in which the height of the seepage face is located as 0.43h, for example, in Figure 17. Yet even in this case the width (b) of the wall is not given, while the phreatic line (free surface curve) can only be located very crudely graphically. Moreover, it would be most helpful to

soils and hydraulic engineers to have the effective conductance G of this wall expressed numerically; that is a numerical value for the coefficient G in the expression:

$$Q = G \cdot k \cdot (Lh) \cdot \frac{h}{b} = G(\frac{kL}{b})h^2$$

where L is the length of the wall, and k is the permeability.

Perhaps it is not realized clearly at the time the solutions are performed that these results, after all, are the very answers engineers are seeking, and such numbers can only be obtained with any precision, from the original calculation sheets. For this reason many workers in the numerical computation field prefer to publish the numerical values of the potential (or stream) function as determined at each point in the grid, together with the final unbalanced residual at each such point.

However, in fairness to the authors, they and their co-workers have done a great service to the profession in publishing, for example, the table of coordinates of the upper and lower nappes corresponding to the relaxation solution for the free overflow from a high, sharp crested, vertical weir.

The writers have taken the liberty of comparing in Table 1 these values with values derived from experimental data by the U.S. Bureau of Reclamation and others as published in the report cited below the table. Over the range of values compared the fit is indeed impressive. If the free surface of the relaxation solution was obtained (or at least refined) independently of this experimental information it constitutes a remarkable testimonial to the value of potential flow methods, in general, and relaxation procedures in particular, in obtaining basic design information for free surface flow where viscous effects are not significant. The writers believe this point might be stressed to the advantage of analytical procedures.

The authors have restricted their attention to the use of uniform square grids or meshes throughout. While this may be justified from the viewpoint of space limitations, it may leave the reader with the impression that such a grid is to be preferred in most cases, which most certainly would be incorrect. Several other mesh systems are preferable for particular instances; the following, among others, are in common use:

(1) Non-uniform rectangular
(2) Oblique
(3) Triangular
(4) Hexagonal

} Meshes

Recently, in connection with analog studies, MacNeal[4] [5] has treated the problem where completely non-uniform grids are employed as sketched in Figure D-1-a. He has proved that the best general values of distribution coefficients for such cases may be obtained by constructing what hydrologists would call a "Thiessen Polygon Network" of tributary influence areas, surrounding each node. (For such Thiessen nets as shown in Figure D-1-b, for example, see p. 276 of the chapter by G. R. Williams on Hydrology, in Reference 7.) From the sides of these polygons and from the distances between nodes, as shown in Figure D-1-c, MacNeal has shown that influence coefficients may be determined, which following the notation of the present paper might be called λ_{ab} where:

$$\lambda_{ab} = d_{ab}/l_{12}$$

WEIR DISCHARGE PROFILE

UPPER NAPPE

x/H	Relaxation (Liu)	Experiment (USBR)	Difference
-2.0	.991	.979	.012
-1.5	.983	.972	.011
-1.0	.965	.958	.007
-0.5	.932	.921	.011
0	.855	.845	.010
+0.5	.678	.670	.008
+1.0	.340	.344	.004
+2.0	-.910	-.905	.005

LOWER NAPPE

x/H	Relaxation (Liu)	Experiment (USBR)	Difference
0	0	0	0
.0625	.068	.066	.002
.125	.091	.096	.005
.250	.113	.112	.001
.375	.101	.102	.001
.500	.065	.070	.005
.625	.010	.021	.011
.750	-.058	-.046	.012
.875	-.135	-.122	.013
1.000	-.222	-.214	.008
1.500	-.705	-.724	.019

Sources:

a. Points computed by Relaxation from original paper.

b. Experimental Points:
USBR Data for $Vo^2/2gh = 0.002$
(Ref. USBR Abstract of "Studies of Crests for Overfall Dams," Bulletin 3, Part VI - Hydraulic Investigations Boulder Canyon Final Reports by J.N. Bradley, Engineer.)

Thus, the equivalent statement to the authors' Equation (5) becomes:

$$\frac{\phi_B}{\lambda_{ab}} + \frac{\phi_C}{\lambda_{ac}} + \frac{\phi_D}{\lambda_{ad}} + \frac{\phi_E}{\lambda_{ae}} + \frac{\phi_F}{\lambda_{af}} - \phi_A\left(\frac{1}{\lambda_{ab}} + \frac{1}{\lambda_{ac}} + \frac{1}{\lambda_{ad}} + \frac{1}{\lambda_{ae}} + \frac{1}{\lambda_{af}}\right) = 0$$

For the special case of square nets, the λ's are all unity, and the above equation reduces to Equation (5).

The foregoing techniques can be used to good advantage both for hand and machine solutions, particularly where irregular boundaries or symmetry conditions are involved. MacNeal points out that all radial angles must be acute if negative resistance is to be avoided in passive electrical resistor networks.

Iteration and relaxation methods demonstrate as intermediate or end results the solution at every point in the net, to give a complete picture of heat or fluid flow or potential function distribution in the region considered. However, in many cases engineers are concerned only with the conditions at one or two points, and the distribution of the solution throughout the remainder of the area is not a matter of prime importance. In such cases, satisfactory solutions using these two methods consume a considerable amount of time in relation to the information desired, and some other approach would prove valuable.

Such an approach is offered by the Monte Carlo method[6], or solution by random walks, in which a method of statistics is applied to the solution of partial differential equations. It is best demonstrated by an example which is given in the Appendix.

APPENDIX

Example of the Monte Carlo Method:

In order to illustrate this method of estimating the stream function or potential function at a point in a fluid flowing irrotationally, a rectangular, two-dimensional bend was chosen as shown in Figure D-2-a. At a distance on each side of the corner equal to the width of the section, the flow was assumed uniform and convenient values were assigned to the stream lines. It will be seen from the solution of the problem that the actual position at which uniform flow is assumed has little effect on the result.

The section is then covered with a square meshwork, of some appropriate size. It will be obvious that the amount of work required for a solution depends on the mesh size chosen, and no method is yet known for determining the optimum size for a particular problem. A convention sketch D-2-b is set up beside the main diagram indicating the direction of travel assigned to each number from 1 to 4. The point P at which the stream function, in this case, is desired is then chosen (it may be noted here that the position of this point may influence the original choice of the net) and can be considered the origin of all movement: a pencil point is placed here.

Briefly, then, the method proceeds as follows. A wooden prism of square cross-section is obtained and the 4 long sides marked 1, 2, 3, 4. The prism is then tossed, as if a die, onto a flat surface, and the number turning up is noted, in order that the pencil may move from P as the number directs. Assume that the number 1 turned up, then the pencil moves to B. The prism is cast again, and 4 appears. The pencil moves to C. Another throw, and 1 appears once more. The pencil is now at D, a point on the boundary, of value 30 units. Consequently in the column marked 30 in the accompanying

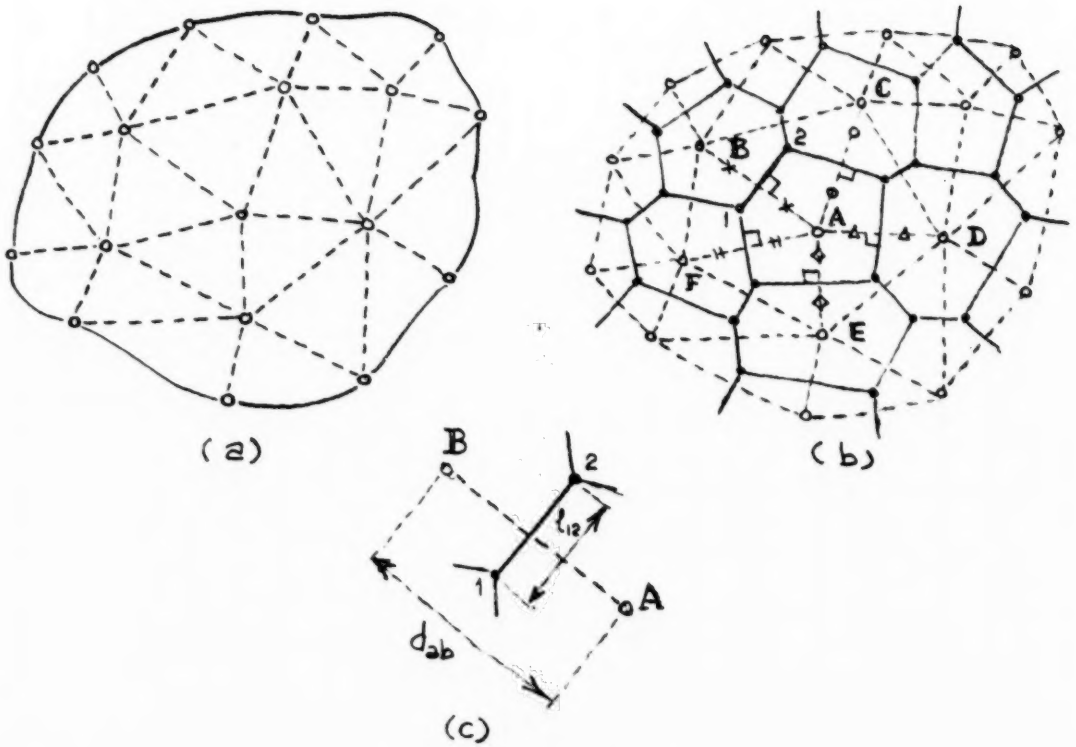
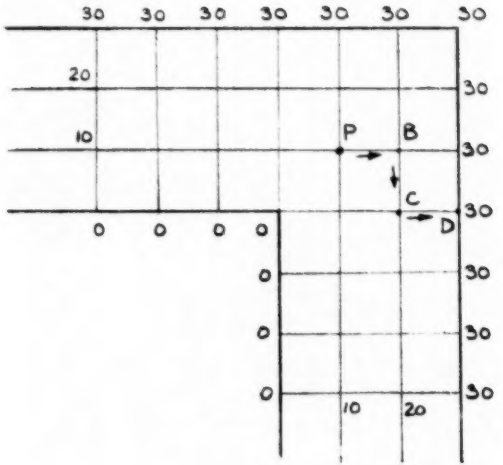


Figure D-1. Irregular Meshes

(a) MESH PATTERN



(b) CONVENTION

(c) TABLE

BOUNDARY	ARRIVALS	TOTAL	COUNT
30	𝍸 𝍸 𝍸

table a check or tally is placed, and the pencil is returned to P. The whole procedure is carried through again, until a boundary is again reached, and the value noted.

Eventually, say, a total of 50 to 100 values have been recorded. A count is made of the number of units from the column corresponding to each boundary point, and these counts are totalled. Division by the number of times boundaries were encountered gives the desired value of the stream function at P.

In the example demonstrated, it is seen that out of a total of 64 boundary arrivals, 43 were on the outer boundary of value 30 units, only one reached the 10 unit interpolated point, and 20 reached the inner 0 boundary. Then, in this case, the value of the stream function at P would amount to a total of (43 x 30) + (1 x 10) + (20 x 0) = 1300 arrival units which, for 64 arrivals gives an average of 1300/64 units = 20.31 units, representing the stream function value at point P.

Instead of a prepared prism, a die may be used, the values 5 and 6 being disregarded, or better still use of a table of random numbers such as is used in statistical work will speed up the work considerably. Any book of mathematical tables in which random digits can occur (for example, certain sections of tables of logarithms) may be utilized. It is not, of course, necessary to use a square mesh, and, if the problem seems to demand it, a triangular hexagonal, or other form may be used, and a suitable convention adopted for the numbering of the directions. This stratagem may be warranted, for example, if a 60^{o} or 120^{o} corner in the boundaries is encountered.

If a three-dimensional problem is to be solved, and a cubical mesh used, the new directions can be assigned the numbers 5 and 6, and a tabular system of coordinates set up, with the values of the various boundaries being noted. Then it would be necessary to check the coordinates of each point arrived at to see if it coincided with a boundary.

From the example demonstrated, it will be seen that the principal contributions to the final result are the values on the two boundaries immediately adjacent to the point, and that the assumption of parallel flow at a section some distance away from the bend plays a very small part in the computations, and this section could be pushed indefinitely outward without materially affecting the labor of solution.

Milne[6] states that the error (correct solution minus approximate solution) in a solution of this sort varies as $1/\sqrt{n}$ where n is the number of arrivals at boundaries, and this fact may be utilized to extrapolate to better solutions. To the writers' knowledge, not enough work has been done with these solution methods as yet to say what is the optimum number of arrivals at boundary points compared to mesh dimensions to obtain reasonably accurate solutions, but further investigation, of a mathematical nature, may point the way.

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*Refers to books.

References (1), (3), (9), and (11) may be recommended as an introduction to methods of relaxation and iteration, and (6) and (8) used more for advanced study.

CORRECTIONS TO PLASTIC BUCKLING OF ECCENTRICALLY LOADED ALUMINUM ALLOY COLUMNS PROCEEDINGS—SEPARATE NO. 299

Page 1. In the first line, the word preceding "load" should be "end" rather than "and".

Page 2. In third line of footnote (2), change "Vol. II" to "Vol. 11".

Page 9. In equation (8), the abbreviation "sec" should be on a level with the horizontal bar in the fraction $\frac{\pi}{2}$, rather than on a level with π.

Page 9. In the first line below equation (8), the word "difference" has been misspelled.

Page 11. In equations (10b) and (10c) the expression immediately preceding the brackets should be $\left(\frac{L}{d}\right)^2$ rather than $\frac{L}{(d)^2}$.

Page 12. In footnote (c) of Table I, the words "1/2-diameter" should be changed to "1/2-in. diameter".

CORRECTIONS TO DESIGNING ALUMINUM ALLOY MEMBERS FOR COMBINED END LOAD AND BENDING PROCEEDINGS—SEPARATE NO. 300

Cover. J. W. Clark should be listed as a Junior Member rather than a Member of ASCE.

Page 9. The figures "10" and "31" in line 10 should be interchanged, so that the sentence becomes, "Equation [10] predicts strengths ranging from 31 per cent above to 10 per cent below the test results".

Page 10. In reference 16, the number "299" should be placed in the blank space following "Separate No."

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